Intensity in single-slit diffraction pattern

We consider a monochromatic light passing through a narrow slit from the left to the right. Huygens' principle implies that we have to consider each point in the slit as a separate source of spherical wavelets propagating in all directions to the right of the slit. The light falls on a screen which is assumed very far away, so the rays heading to any point are nearly parallel. We expect interference from waves originated at different points of the slit.

Here we will discuss a method of predicting the amplitude and intensity of the light waves at any point on the interference pattern using the phasor technique that is a little bit different than in the text-book. We will find the amplitude and intensity of the wave created by interference of large number of waves originated at the slit at some point P at the screen. The position of the point P is characterized by its angular position θ.
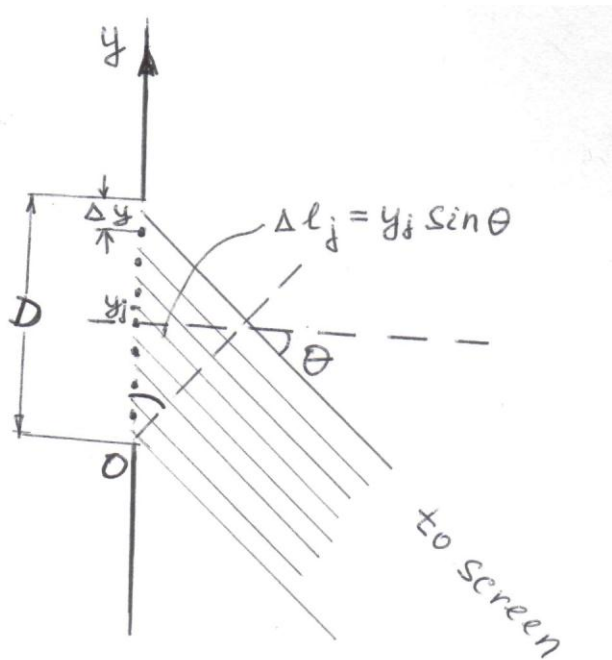

Fig. 1

Let us divide the slit (see Fig. 1) into very large number N of very thin strips of equal size (the width) Δy. We assume that all waves (rays) from a given strip are in phase. However, waves from different strips are different in phase. Let us choose the y-axis as

shown in Fig. 1, so that the bottom edge of the slit has y-coordinate equal 0 and the top edge has y-coordinate equal D. We will characterize the phase of waves from a given j-strip due to extra distance traveled by these waves in comparison with waves from the bottom strip by

$$\Delta\beta_j = k\Delta l_j = \frac{2\pi}{\lambda} y_j \sin\theta \qquad (1)$$

Here $k = 2\pi/\lambda$ is the wave number, Δl_j is the extra-distance traveled to the point P at the screen by rays from the j-strip, y_j is the coordinate of a middle-point of the j-strip.

The electric field $\vec{E}$ at the point P is sum of electric field vectors of waves coming from the slit. We will assume that all coming waves have parallel electric fields with the same amplitude E_0. Then the wave created at the point P as result of interference can be written in the form

$$E_\theta(t) = \sum_{j=1}^{N} E_0 \cos(\omega t - kl_0 - \Delta\beta_j)\frac{\Delta y}{D} \qquad (2)$$

One can easy find from (2) the phasor of the sinusoidal wave E_θ

$$\underline{E_\theta} = \sum_{j=1}^{N} E_0 e^{-ikl_0 - i\Delta\beta_j}\frac{\Delta y}{D} = E_0 e^{-ikl_0}\sum_{j=1}^{N} e^{-i\Delta\beta_j}\frac{\Delta y}{D} \qquad (3)$$

We calculate (3) by using the limit $N \to \infty$ and $\Delta y \to 0$ and by replacing the sum over j by integral over y. As a result, we obtain

$$\underline{E_\theta} = E_0 e^{-ikl_0}\frac{1}{D}\int_0^D e^{-i\Delta\beta(y)}dy \qquad (4)$$

where according to Eq. (1)

$$\Delta\beta(y) = \frac{2\pi}{\lambda} y \sin\theta \qquad (5)$$

Substituting (5) into (4) we obtain

$$\underline{E}_\theta = E_0 e^{-ikl_0} \frac{1}{D} \int_0^D e^{-i\frac{2\pi}{\lambda}y\sin\theta} dy = E_0 e^{-ikl_0} \frac{1}{D} i \frac{\lambda}{2\pi\sin\theta} \left(e^{-i\frac{2\pi}{\lambda}D\sin\theta} - 1 \right)$$

$$= E_0 e^{-ikl_0} \frac{1}{D} i \frac{\lambda}{2\pi\sin\theta} e^{-i\frac{\pi}{\lambda}D\sin\theta} \left(e^{-i\frac{\pi}{\lambda}D\sin\theta} - e^{i\frac{\pi}{\lambda}D\sin\theta} \right) \qquad (6)$$

$$= E_0 e^{-ikl_0} e^{-i\frac{\pi}{\lambda}D\sin\theta} \frac{\lambda}{\pi D\sin\theta} \sin\frac{\pi D\sin\theta}{\lambda}$$

Thus, the amplitude of the wave created as a result of interference at the point P is equal

$$E_\theta = E_0 \frac{\sin\dfrac{\pi D\sin\theta}{\lambda}}{\dfrac{\pi D\sin\theta}{\lambda}} \qquad (7)$$

Intensity of the wave is proportional to the square of the wave amplitude

$$I_\theta = CE_0^2 \left(\frac{\sin\dfrac{\pi D\sin\theta}{\lambda}}{\dfrac{\pi D\sin\theta}{\lambda}} \right)^2 ; \quad C = \text{const} \qquad (8)$$

One can see from Eq. (8) that intensity has its maximum at $\theta = 0$ when expression in brackets in (8) is equal 1. The value of intensity is

$$I_0 = CE_0^2 \qquad (9)$$

Thus, the wave intensity at the point with angular position θ has the form

$$I_\theta = I_0 \left(\frac{\sin\dfrac{\pi D\sin\theta}{\lambda}}{\dfrac{\pi D\sin\theta}{\lambda}} \right)^2 \qquad (10)$$

Minima of intensity occur when $\sin(\pi D\sin\theta/\lambda) = 0$, that means that

$$D\sin\theta = m\lambda, \quad m = \pm 1, \pm 2 \dots$$

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